The statement was made during lecture that the uncertainty principle expression implies quantum mechanical kinetic energy is of the form:

\[ E \propto \frac{h^2}{2m\Delta x^2} \]

While this is obviously true for the particle in an infinite square well (particle in a box) and for the rigid rotor, textbook expressions for the energy of the H atom and one-electron ions are most always expressed in terms a different characteristic length, \( a \), the Bohr radius, and the mass and Planck’s constant do not appear:

\[ E = -\frac{Z^2}{2n^2} \frac{e^2}{4\pi \varepsilon_0 a}, \text{ in SI units, where } a \text{ stands for the collection of constants } \frac{4\pi \varepsilon_0 h^2}{\mu e^2} \]

\( \mu \) is the reduced mass of the nucleus-electron pair, and \( e \) is the proton charge.

The harmonic oscillator energy levels are almost always expressed as \( E_n = (n + 1/2) \hbar \omega \), but can be expressed in terms of a common characteristic length for the harmonic oscillator, the classical turning point in the zero point state, \( x_0 = \left( \frac{\hbar}{\mu \omega} \right)^{1/2} \), which is most often seen in the harmonic oscillator wave function as: \( \alpha = \left( \frac{\mu \omega}{\hbar} \right) = \frac{1}{x_0^2} \)

It is possible to write an exact energy level expression for any system in the form:

\[ E \propto \frac{h^2}{2m\Delta x^2} \]

Write the expression for the zero point energy in this form for the following systems:
(a) particle in a box
(b) rigid rotor
(c) hydrogen atom
(d) harmonic oscillator